Regional Mathematical Olympiad – 2023

Time: 3 hours **October 29, 2023**

Instructions

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let N be the set of all positive integers and $S=\{(a,b,c,d)\in \mathrm{N}^4 ; a^2+b^2+c^2=d^2\}$. Find the largest positive integer m such that m divides $abcd$ for all $(a,b,c,d)\in S$.

2. Let ω be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C, D lie in the interior of the arc AB. Let E be a point on the diameter AB such that CE and DE are equally inclined to the line AB. Prove that:

(a) The measure of $\angle CED$ is a constant.

(b) The circumcircle of triangle CED passes through a fixed point.

3. For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m<n$ and

$(s(n))^2=m$ and $(s(m))^2=n$.

4. Let Ω_1, Ω_2 be two intersecting circles with centres O_1, O_2 respectively. Let l be a line that intersects Ω_1 at points A, C and Ω_2 at points B, D such that A, B, C, D are collinear in that order. Let the perpendicular bisector of segment AB intersect Ω_1 at points P, Q; and the perpendicular bisector of segment CD intersect Ω_2 at point R, S such that P, R are on the same side of l. Prove that the midpoint of PR, QS and $O_1\,O_2$ are collinear.

5. Let $n>k>1$ be positive integers. Determine all positive real numbers $a_1, a_2, ..., a_n$ which satisfy

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)\,a_i^k+1}}=\sum_{i=1}^{n}a_i=n.$$

6. Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.